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Reflection of a shallow-water soliton. Part 1. Edge layer for shallow-water waves

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To investigate reflection of a shallow-water soliton at a sloping beach, the edge-layer theory is developed to obtain a 'reduced' boundary condition relevant to the simplified shallow-water equation describing the weakly dispersive waves of small but finite amplitude. An edge layer is introduced to take account of the essentially two-dimensional motion that appears in the narrow region adjacent to the beach. By using the matched-asymptotic-expansion method, the edge-layer theory is formulated to cope with the shallow-water theory in the offshore region and the boundary condition at the beach. The 'reduced' boundary condition is derived as a result of the matching condition between the two regions. An explicit edge-layer solution is obtained on assuming a plane beach.

1. Introduction

Reflection of a shallow-water soliton at a sloping beach is interesting not only from the standpoint of coastal engineering, but also with regard to the current soliton research being vigorously investigated in many fields. The main concern in coastal engineering is to understand the nearshore behaviour of wave motions; in particular to estimate the maximum run-up on the beach when tsunamis are incident. Tsunamis are usually treated as bores, for which the propagation of a discontinuity is discussed using the nonlinear shallow-water equation of hyperbolic type (Keller, Levine & Whitham 1960; Meyer & Taylor 1972; Whitham 1979; Hibberd & Peregrine 1979). The run-up of solitary waves on a beach has also been investigated by several authors (Peregrine 1967; Pedersen & Gjevik 1983; Kim, Liu & Liggett 1983). On the other hand, little progress has been made on the reflection problem in recent soliton research. Most work has been concerned with an infinitely extending space region. Exceptionally, however, reflection at a vertical rigid wall has been studied, because the mirror-image principle makes it possible to remove the wall in order to replace the problem by a collision of two identical solitons in an infinite region (Oikawa & Yajima 1973; Miles 1977*a*,*b*; Funakoshi & Oikawa 1982). Hence the purpose of this series of papers is to aim at reflection of a shallow-water soliton incident upon a sloping beach. To this end, we develop in this paper the edge-layer theory to derive a 'reduced' boundary condition relevant to the simplified equation for shallow-water waves of small but finite amplitude in the offshore region.

Suppose a soliton to be incident from the infinity to a sloping beach whose surface has a general form, as depicted in figure 1. It should be remarked that the incident wave is not necessarily a soliton, but it may be a continuous wavetrain. It is only assumed that the well-known weakly nonlinear and dispersive wave theory (e.g. Whitham 1974) is applied to the offshore region. For a short while, until the soliton

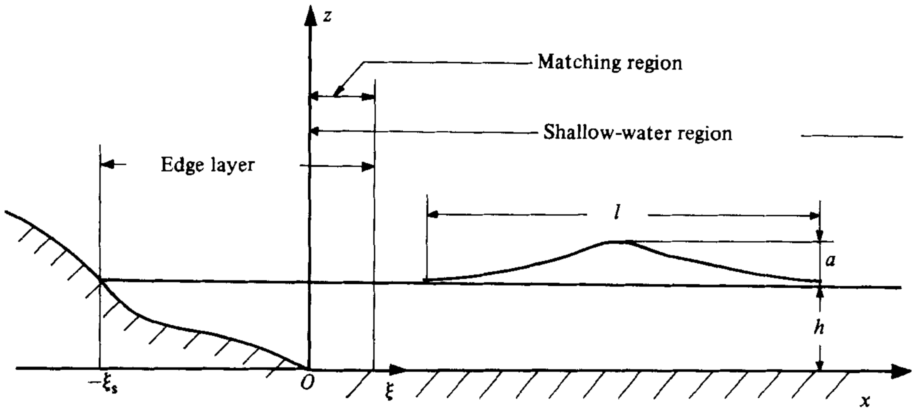

FIGURE 1. Geometrical configuration of a reflection problem.

approaches near the shore, its propagation is not affected by the existence of the beach. The fluid motion is then almost in the horizontal direction. As it approaches near to the shore, however, the boundary condition at the beach requires that the normal velocity component to the beach surface vanishes, so that the fluid motion is obliged to be essentially two-dimensional. Accordingly the shallow-water theory becomes invalid in the vicinity of the beach. But as it propagates back seaward after reflection at the beach, it is expected on physical grounds that the shallow-water theory resumes its validity in the offshore region. Thus in the vicinity of the shore, we introduce a narrow region called an edge layer† in which the two-dimensional motion is of primary importance. Over this layer, the shallow-water theory is bridged to the boundary condition at the beach surface.

In §2 we recapitulate to the shallow-water theory, in which a balance between both weak effects of nonlinearity and dispersion is assumed. Then the wave behaviour is described by a simplified equation similar in form to the Boussinesq equation. For the edge layer, a reformulation is made in §3 so as to take account of the fully two-dimensional motion. Here it is noted that the edge-layer equations are linear because as far as a balance of the two aforementioned effects in the shallow-water theory is maintained, the width of the edge layer is too narrow for the effect of nonlinearity to accumulate over this layer. On assuming such a narrow edge layer, the matched-asymptotic-expansion method (e.g. Cole 1968) is employed to obtain the edge-layer solution. From the matching condition between the shallow-water region and the edge layer, we derive in §4 the 'reduced' boundary condition relevant to the simplified shallow-water equation mentioned above. By solving such a boundary-value problem, reflection of a soliton is expected to be clarified. Of course, this condition should not be applied at the actual beach but in the vicinity of the beach in some sense. The 'reduced' boundary condition depends only on the average inclination of the beach surface, not on its local form. This suggests that the beach surface may be substantially replaced by a plane beach as far as the 'reduced' boundary condition is concerned. In the edge layer, on the other hand, the precise form of the beach surface is naturally of essential importance. It is found that the time dependence of the edge layer appears parametrically only through the scale factors to be determined

† This is a sort of boundary layer in a wide sense. But since no viscous effect is involved, the term 'edge layer' is used to avoid confusion with the ordinary boundary layer. Phenomena similar to this edge layer are investigated in the flexural motions of a thin elastic plate (Sugimoto 1981 *a*, *b*).

after solving a reflection problem in the shallow-water region. In this sense the edge-layer solution represents a standing wave. A limiting case is discussed in which the beach is no longer confined to a narrow region but to a region comparable to one wavelength of shallow-water waves. In this case, there might be an alternative way to employ the shallow-water theory everywhere, including the beach. Even in such a case, however, it is shown that the present results can be applied with some modifications. As a simple but typical example, we obtain in §5 an edge-layer solution for a plane beach. Although the solution is expressed analytically in an integral form, its explicit evaluation is left for a future work.

2. Summary of shallow-water theory

In this section the results of shallow-water theory are briefly summarized. For the detailed exposition, reference should be made to Whitham (1974). Two-dimensional and irrotational motion of an inviscid fluid of constant depth is governed by the Laplace equation together with boundary conditions at the bottom and the free surface:

$$\beta\phi_{xx}+\phi_{zz}=0 \quad \text{for} \quad 0<z<1+\alpha\eta, \tag{2.1}$$

$$\phi_z=0 \quad \text{at} \quad z=0, \tag{2.2}$$

$$\left.\begin{aligned} &\phi_z-\beta\eta_t-\alpha\beta\phi_x\eta_x=0, &(2.3a)\\ &\eta+\phi_t+\tfrac{1}{2}\alpha\phi_x^2+\frac{\alpha}{2\beta}\phi_z^2=0 &(2.3b)\end{aligned}\right\} \quad \text{at} \quad z=1+\alpha\eta,$$

where x, z and t are the horizontal and vertical coordinates and the time, normalized respectively by a characteristic wavelength l, the depth h, and $l/(gh)^{\frac{1}{2}}$, g being the acceleration due to gravity. $\phi(x,z,t)$ and $\eta(x,t)$ denote the velocity potential and the surface elevation from the quiescent level, which are normalized respectively, by $agl/(gh)^{\frac{1}{2}}$ and a, a being a characteristic elevation; the subscripts imply partial differentiation. Here two parameters $\alpha(=a/h)$ and $\beta(=(h/l)^2)$ imply the degree of finite amplitude and that of shallow water. In the following analysis, they are assumed to be small and of the same order of magnitude ($\alpha\sim\beta\ll 1$).

Making use of the two small parameters thus introduced, the solution of (2.1) with (2.2) can be expressed as

$$\phi(x,z,t)=\sum_{n=0}^{\infty}\left(-\beta\frac{\partial^2}{\partial x^2}\right)^n\frac{f(x,t)\,z^{2n}}{(2n)!}. \tag{2.4}$$

Substitution of (2.4) into (2.3) and elimination of η yield

$$\eta=-f_t-\tfrac{1}{2}\alpha f_x^2+\tfrac{1}{2}\beta f_{ttt}+O(\alpha\beta,\beta^2), \tag{2.5}$$

$$f_{tt}-f_{xx}-\tfrac{1}{3}\beta f_{xxtt}=-\alpha(\tfrac{1}{2}f_t^2+f_x^2)_t+O(\alpha\beta,\beta^2). \tag{2.6}$$

These are the well-known results in the shallow-water theory. From (2.6), f, the lowest term of the velocity potential ϕ, is governed by the equation similar to the usual Boussinesq equation derived for a surface elevation. As has already been pointed out (see Miles 1980), however, the Boussinesq equation fails to describe correctly bidirectional wave propagation, because the nonlinear term is simplified by using the assumption for the unidirectional wave, i.e. $f_t\sim -f_x$. In the following analysis, therefore, (2.6) is employed instead of the Boussinesq equation. In addition to this, we should note the form of the highest derivative f_{xxtt}. By using the relation $f_{tt}-f_{xx}=O(\alpha,\beta)$, f_{xxtt} can equivalently be written as f_{xxxx} or f_{tttt} within the lowest approximation. As far as an infinitely extending space region is concerned, such a replacement does not produce any substantial difference physically, although it is mathematically different. But if a boundary-value problem with respect to x is concerned, a relevant form should be selected so as to be consistent with a number of boundary conditions. As we shall see later, one and only one boundary condition can be imposed for f at a beach, and therefore f_{xxtt} is relevant.

3. Formulation of edge-layer theory

In §1 we have described physically the necessity of introducing an edge layer in the shallow-water theory when a beach is present. We now estimate a lengthscale over which the edge layer develops seaward. To this end, the linear theory gives a useful estimation as the lowest approximation. The linearized wave solution of (2.1)–(2.3) has two modes. One is the usual propagating mode for which the potential ϕ is given by $\phi \propto \cosh(\beta^{\frac{1}{2}}kz)\exp[\mathrm{i}(kx-\omega t)]$, where the wavenumber k and frequency ω satisfy the dispersion relation $\beta^{\frac{1}{2}}\omega^2 = k\tanh(\beta^{\frac{1}{2}}k)$. The other is the evanescent mode, which is confined to the neighbourhood of obstacles or boundaries. This mode has a potential of the form $\phi \propto \cos(\beta^{\frac{1}{2}}kz)\exp(-kx-\mathrm{i}\omega t)$ $(k>0)$, whose dispersion relation is given by $\beta^{\frac{1}{2}}\omega^2 = -k\tan(\beta^{\frac{1}{2}}k)$. For a frequency $\omega(=O(1))$ of the incident wave, the positive wavenumber takes $\beta^{\frac{1}{2}}k \approx \pi, 2\pi, \ldots$. This shows that the evanescent mode has a local influence over the uniform bed to a distance $x \approx 1/k \approx \beta^{\frac{1}{2}}/\pi \ll 1$ (which will be confirmed later by the edge-layer solution (5.19a)). Hence the edge layer appears in the vicinity of $x=0$, in which the two-dimensional motion is seen to be essential from the structure of ϕ. For a proper description of this layer, we stretch the coordinate x near $x=0$ in terms of a new variable $\xi = x/\beta^{\frac{1}{2}}$. The introduction of ξ implies the renormalization of the horizontal scale x by h. Except for x, the other quantities are used as defined before. In particular, it should be noted that time t is still normalized by $l/(gh)^{\frac{1}{2}}$ not by $h/(gh)^{\frac{1}{2}}$. This means that the timescale in the edge layer is still assumed to be characterized by that in the shallow-water region.

Changing the variable from x to ξ in (2.1)–(2.3), the basic system of equations is rewritten as

$$\phi_{\xi\xi}+\phi_{zz} = 0 \quad \text{for} \quad 0<z<1+\alpha\eta, \tag{3.1}$$

$$\phi_z = 0 \quad \text{at} \quad z=0, \tag{3.2}$$

$$\left.\begin{aligned} \phi_z-\beta\eta_t-\alpha\phi_\xi\eta_\xi &= 0, \quad (3.3a)\\ \eta+\phi_t+(\alpha/2\beta)(\phi_\xi^2+\phi_z^2) &= 0 \quad (3.3b)\end{aligned}\right\} \quad \text{at} \quad z=1+\alpha\eta,$$

where ϕ is now a function of ξ, z and t, while η is a function of ξ and t. In addition to (3.2) and (3.3), the boundary condition at the beach should be imposed explicitly in the edge-layer theory. Let the beach surface be given by the general form $\xi = b(z)$ $(-\xi_s < \xi < 0)$, where $-\xi_s$ is the position of the shoreline.† Then the boundary condition takes the form

$$v_n = -\frac{\beta^{-\frac{1}{2}}}{(1+b_z^2)^{\frac{1}{2}}}(\phi_\xi - b_z\phi_z) = 0 \quad \text{at} \quad -\xi_s < \xi = b(z) < 0, \tag{3.4}$$

where v_n denotes the velocity component along the outward normal to the beach surface. In light of the normalization of the velocity potential, note that $\beta^{-\frac{1}{2}}\phi_\xi$ and

† Here we assume a one-to-one correspondence between ξ $(=b(z))$ and z, namely that $\xi = b(z)$ is a monotonically decreasing function of z.

$\beta^{-\frac{1}{2}}\phi_z$ represent respectively the velocity components along the ξ- and z-directions, normalized by $ag/(gh)^{\frac{1}{2}}$. On the other hand, the edge layer is assumed to vanish seaward, so that the shallow-water region reappears in the region $x = O(1)$. In the overlapping region between them, ϕ and η in the edge layer should match smoothly with those in the shallow-water region in the vicinity of $x = 0$. Following the idea of the matched-asymptotic-expansion method (e.g. Cole 1968), such a matching region corresponds to $\xi \to \infty$ but still $x = 0$. Thus the matching conditions for ϕ and η at $\xi \to \infty$ are obtained by expanding (2.4) and (2.5) around $x = 0$ and later replacing x by $\beta^{\frac{1}{2}}\xi$. Denoting them by ϕ_∞ and η_∞ respectively, we have

$$\phi_\infty = [f+\beta^{\frac{1}{2}}f_x\xi+\tfrac{1}{2}\beta f_{xx}(\xi^2-z^2)]_{x=0}+O(\beta^{\frac{3}{2}}), \tag{3.5a}$$

$$\eta_\infty = [-f_t-\beta^{\frac{1}{2}}f_{xt}\xi-\tfrac{1}{2}\alpha f_x^2+\tfrac{1}{2}\beta(f_{ttt}-f_{xxt}\xi^2)]_{x=0}+O(\alpha\beta^{\frac{1}{2}},\beta^{\frac{3}{2}}), \tag{3.5b}$$

where f, f_x, f_{xx}, etc. are evaluated at $x = 0$, and therefore they are functions of t only. Hereinafter the symbol $[\ldots]_{x=0}$ implying evaluation at $x = 0$ will be omitted for simplicity. Hence the matching conditions for ϕ and η take the forms

$$\phi \to \phi_\infty \quad \text{and} \quad \eta \to \eta_\infty \quad \text{as} \quad \xi \to \infty. \tag{3.6}$$

Equation (3.1) together with (3.2)–(3.6) constitutes the boundary-value problem for the edge layer.

In the light of (3.6), let us introduce ψ defined as

$$\phi = \phi_\infty + 2\beta^{\frac{1}{2}}f_x\psi, \tag{3.7}$$

where ψ is an unknown function of ξ, z and t which represents the deviation of the edge-layer solution from the shallow-water theory owing to the presence of the beach. The factor $\beta^{\frac{1}{2}}$ is suggested from (3.5a) and the condition (3.11) appearing later, because if it were not introduced $\beta^{\frac{1}{2}}$ would appear on the right-hand side of (3.11). Since ϕ_∞ is evaluated up to $O(\beta)$ in (3.5a), ψ should be evaluated up to $O(\beta^{\frac{1}{2}})$. Upon substituting (3.7) into (3.1), ψ must satisfy the Laplace equation

$$\psi_{\xi\xi}+\psi_{zz} = 0 \tag{3.8}$$

and the boundary condition at the bottom surface

$$\psi_z = 0 \quad \text{at} \quad z = 0. \tag{3.9}$$

To impose the boundary conditions at the free surface, the small parameter α enables us to employ them in the expanded form around $z = 1$:

$$\left.\begin{aligned} \psi_z + O(\beta) &= 0, \\ \eta &= \eta_\infty - 2\beta^{\frac{1}{2}}(f_x\psi)_t + O(\beta) \end{aligned}\right\} \quad \text{at} \quad z = 1, \tag{3.10a, b}$$

where we have discarded the nonlinear terms $O(\alpha)$ as well as $O(\beta)$, since we are concerned with ψ up to $O(\beta^{\frac{1}{2}})$.

The boundary condition at the beach is now written as

$$2f_x(\psi_\xi - b_z\psi_z) = -\left\{f_x+\beta^{\frac{1}{2}}f_{xx}\frac{\mathrm{d}}{\mathrm{d}z}(zb)+\tfrac{1}{2}\beta f_{xxx}\left[\frac{\mathrm{d}}{\mathrm{d}z}(zb^2)-z^2\right]\right.$$
$$\left. +O(\beta^{\frac{3}{2}})\right\} \quad \text{at} \quad -\xi_s < \xi = b(z) < 0. \tag{3.11}$$

On the other hand, owing to (3.7), the matching condition (3.6) simply assumes the form

$$\psi \to 0 \quad \text{as} \quad \xi \to \infty. \tag{3.12}$$

It is thus found that the problem of finding ψ is reduced to solving the Laplace equation (3.8) under the boundary conditions (3.9), (3.10a), (3.11) and (3.12); that is, to finding an irrotational flow field in the semi-infinite channel region caused by the source and sink distributed over the boundary at $\xi = b(z)$. After solving ψ, the elevation η is obtained from (3.10b). As will be shown in §4, the condition (3.11) is replaced by (4.4), which does not involve t explicitly, so that ψ is independent of t. Hence the time dependence of ϕ and η in the edge layer appears parametrically only through the factors f_x, f_{xx}, and so on at $x = 0$, which are determined after solving a reflection problem in the shallow-water region. In this sense, the edge-layer solution represents a standing wave.

4. Reduced boundary condition

Before seeking the edge-layer solution, we consider the boundary condition (3.11) and the matching condition (3.12). From the fluid-dynamical interpretation, (3.11) gives the strength of source and sink distributed over the boundary, while (3.12) indicates no fluid flux from an infinite open end. Since there is no other source or sink in the flow field, a compatibility condition must be imposed on (3.11) to guarantee mass conservation in the edge layer. Indeed, integrating (3.8) over the edge-layer region and applying Stokes' theorem, we find that the following line integral along the boundary ∂S of the region concerned must vanish:

$$\oint_{\partial S} (\psi_\xi \,\mathrm{d}z - \psi_z \,\mathrm{d}\xi) = 0. \tag{4.1}$$

Noting (3.9), (3.10a), and (3.12), and using (3.11), it follows from (4.1) that

$$\frac{1}{f_x}\left[f_x \int_b \mathrm{d}z + \beta^{\frac{1}{2}} f_{xx} \int_b \mathrm{d}(zb) + \tfrac{1}{2}\beta f_{xxx} \int_b \mathrm{d}(zb^2 - \tfrac{1}{3}z^3)\right] + O(\beta) = 0, \tag{4.2}$$

where $\int_b$ denotes integration along the sloping beach $b(z)$, and $f_x \neq 0$ has been assumed. Further noting that the depth is unity and that $[zb]_{z=1} = -\xi_s$ and $f_{xxx} = f_{xtt} + O(\alpha, \beta)$ from (2.6), we have

$$f_x = \beta^{\frac{1}{2}} \xi_s f_{xx} + O(\beta^{\frac{3}{2}}). \tag{4.3}$$

Substituting this into (3.11), we obtain simply

$$\psi_\xi - b_z \psi_z = -\frac{1}{2}\left[\frac{1}{\xi_s}\frac{\mathrm{d}}{\mathrm{d}z}(zb) + 1\right] + O(\beta) \quad \text{at} \quad -\xi_s < \xi = b(z) < 0. \tag{4.4}$$

On integrating (4.4) over the depth, it is indeed found that the total fluid flux across the boundary cancels out. In addition to such mass conservation, (4.3) is interpreted differently as follows. Remembering that f_x and f_{xx} are evaluated at $x = 0$, (4.3) is regarded as a 'reduced' boundary condition for (2.6) at $x = 0$. By the term 'reduced' we mean that (4.3) is derived by averaging the effect of edge layer to recast the exact boundary condition into a form relevant to (2.6). By solving (2.6) under (4.3), it is expected that the behaviour of shallow-water waves during reflection at the sloping beach is clarified. After completing such an 'exterior' problem, the time-dependent factors f_x, f_{xx}, and so on at $x = 0$ can be specified. The exterior problem is left for a forthcoming paper. Here it is emphasized that (4.3) is no longer applied at the beach in the exact sense, but in the vicinity of the beach $x = 0$.† Also it is emphasized that

† By invoking Taylor's theorem, (4.3) is formally regarded as the first two terms of $f_x(x = -\beta^{\frac{1}{2}}\xi_s) = 0$, though $x = -\beta^{\frac{1}{2}}\xi_s$ is located outside the region of validity of (2.6). It is of interest to note that this corresponds to extending the region of constant depth to the shoreline and applying $f_x = 0$ at that point.

(4.3) depends only on ξ_s, not on the local form of the beach surface. Noting again that the depth is unity, ξ_s represents the inverse of the inclination of the chord connecting the coordinate origin and the shoreline to the horizontal. In other words, as far as (4.3) is concerned, a beach surface of general form may be replaced by a plane beach with constant inclination ξ_s^{-1}.

Next we discuss the magnitude of inclination ξ_s^{-1} of the beach surface. In the very special case with a vertical plane wall ($b = 0$), (4.3) is reduced to the usual boundary condition $f_x = 0$, as expected. There then arises no need to introduce ψ, because (3.11) is automatically satisfied. In this case, no edge layer appears and the shallow-water theory can be applied up to the 'beach'. This is physically obvious from the mirror-image principle. Incidentally, if a soliton is incident on such a vertical wall, it is well known that the reflected soliton undergoes only a phase shift $O(\beta)$ (Oikawa & Yajima 1973; Miles 1977*a*; Funakoshi & Oikawa 1982). For small values of ξ_s^{-1}, on the other hand, ξ_s^{-1} has the lower limit $\xi_s^{-1} = O(\beta^{\frac{1}{2}})$, which can easily be understood from the scaling discussion given in §3. For such a limit case, the variation of the beach surface is so gentle that the width of the edge layer becomes comparable to one characteristic wavelength. Even in this special case, the results (4.3) and (4.4) can still be applied with some modifications. To specify a gentle slope $O(\beta^{\frac{1}{2}})$, we assume that a beach surface is expressed by $\xi = \beta^{-\frac{1}{2}}b_0(z)$, where $b_0(z)$ is $O(1)$. Since ξ is subject to a variation $O(\beta^{-\frac{1}{2}})$, we must start again with evaluating ϕ_∞ in (3.5). From (2.4), ϕ_∞ is now expressed as

$$\phi_\infty = \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{p!\,(2n)!}\beta^{\frac{1}{2}p+n}\left[\frac{\partial^{p+2n}f}{\partial x^{p+2n}}\right]_{x=0}\xi^p z^{2n}. \tag{4.5}$$

Substitution of (3.7) with (4.5) into (3.4) yields

$$2f_x(\psi_\xi - b_z\psi_z) = \sum_{n=0}^{\infty}\beta^n\frac{(-1)^{n+1}}{(2n+1)!}\left\{\sum_{p=0}^{\infty}\frac{1}{p!}\frac{\partial^p}{\partial x^p}\left[\frac{\partial^{2n+1}f}{\partial x^{2n+1}}\right]_{x=0}\frac{\mathrm{d}}{\mathrm{d}z}(z^{2n+1}b_0^p)\right\}$$
$$\text{at}\quad -\xi_s = -\beta^{-\frac{1}{2}}\xi_0 < \xi = \beta^{-\frac{1}{2}}b_0(z) < 0, \tag{4.6}$$

where $\xi_0 = -b_0(1)$. Taking the same line integral as in (4.1), it follows from the lowest-order terms that

$$\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\frac{\partial^p}{\partial x^p}\left(\frac{\partial f}{\partial x}\right)\xi_0^p + O(\beta) = 0 \quad \text{at} \quad x = 0, \tag{4.7}$$

from which the right-hand side of (4.6) takes the form

$$\sum_{p=1}^{\infty}\frac{1}{p!}\frac{\partial^p}{\partial x^p}\left(\frac{\partial f}{\partial x}\right)\left[(-\xi_0)^p - \frac{\mathrm{d}}{\mathrm{d}z}(zb_0^p)\right] + O(\beta). \tag{4.8}$$

Hence the condition (4.3) is now replaced by (4.7). In connection with the footnote concerning (4.3), we only note that (4.7) is formally expressed as $f_x(x = -\xi_0) = 0$. Also the condition (4.7) can alternatively be written in terms of f_x and f_{xx} only. Using (2.6), i.e. $f_{tt} - f_{xx} = O(\alpha, \beta)$, and noting that

$$f_{xxx} = f_{xtt} + O(\alpha,\beta), \quad f_{xxxx} = f_{xxtt} + O(\alpha,\beta), \quad \text{etc.}, \tag{4.9}$$

(4.7) can be expressed as

$$\mathrm{L}_1\left(\frac{\partial f}{\partial x}\right) = \mathrm{L}_2\left(\frac{\partial^2 f}{\partial x^2}\right), \tag{4.10a}$$

with the operators L_1 and L_2 defined by

$$L_1 = 1+\frac{\xi_0^2}{2!}\frac{\partial^2}{\partial t^2}+\frac{\xi_0^4}{4!}\frac{\partial^4}{\partial t^4}+\dots \tag{4.10b}$$

and

$$L_2 = \xi_0+\frac{\xi_0^3}{3!}\frac{\partial^2}{\partial t^2}+\frac{\xi_0^5}{5!}\frac{\partial^4}{\partial t^4}+\dots. \tag{4.10c}$$

Applying the Fourier transform defined by

$$\mathscr{F}\left(\frac{\partial f}{\partial x}\right) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{\partial f(0,t)}{\partial x}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \tag{4.11}$$

to (4.10a), it follows that

$$\mathscr{F}\left(\frac{\partial f}{\partial x}\right) = \frac{1}{\omega}\tan(\xi_0\omega)\,\mathscr{F}\left(\frac{\partial^2 f}{\partial x^2}\right). \tag{4.12}$$

By the inverse transform, (4.12) is expressed as

$$\frac{\partial f}{\partial x} = \xi_0\frac{\partial^2 f}{\partial x^2}-\frac{\xi_0^3}{3}\frac{\partial^4 f}{\partial t^2\,\partial x^2}+\dots+O(\beta), \tag{4.13a}$$

or alternatively

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\xi_0}\frac{\partial f}{\partial x}+\frac{\xi_0}{3}\frac{\partial^3 f}{\partial t^2\partial x}+\dots+O(\beta), \tag{4.13b}$$

which is the 'reduced' boundary condition for the case with $\xi_s^{-1} = O(\beta^{\frac{1}{2}})$. It should be remarked, however, that (4.13) contains the error $O(\beta)$ while (4.3) contains one $O(\beta^{\frac{3}{2}})$. In this sense, (4.13) is regarded as an approximate boundary condition to (2.6) to the lowest order. It should also be noted that (4.13) involves the additional terms, and some appropriate truncation is necessary for practical use. There are two possibilities which allow the truncation by a few terms. One is the case for which ξ_0 is sufficiently small. This is nothing but the case already treated ($O(\beta^{\frac{1}{2}}) < \xi_s^{-1} = O(1)$), for which (4.13) is reduced to (4.3). The other is the case for which the temporal variation of f is fairly slow, so that the higher-order derivatives of f with respect to t become small. This imposes some restriction on the incident waves. In this connection we should mention the following point. For a gentle slope with the inclination $O(\beta^{\frac{1}{2}})$, an 'additional' elevation of the free surface in the edge layer may occur by the shoaling process as a wave approaches the shoreline. Then there arises some concern as to whether the nonlinear terms neglected in (3.10) become comparable to the retained terms. But it is found from the linear theory that the increases in the elevation and wave slope remain within a variation of order unity. To see this, we note that the matching region is located at a distance $\xi_\infty = O(\beta^{-\frac{1}{2}})$ from the shoreline, so that ξ_∞/ξ_s is of order unity. From the 'shallow-water approximation' for a plane beach with an inclination $\theta(= O(\beta^{\frac{1}{2}}) \ll 1)$, it is known that the ratio of the amplitude a_∞ at the matching region to a_0 at the shoreline is given, for a perfect reflection, by $J_0[2(\beta\xi_\infty/\theta)^{\frac{1}{2}}\omega]$, where J_0 is the Bessel function of order zero and $\omega(= O(1))$ is a frequency of the incident waves (cf. (7.45) in Whitham 1979). Here we should emphasize again that the timescale in the edge layer is $l/(gh)^{\frac{1}{2}}$ not $h/(gh)^{\frac{1}{2}}$. Hence the increase in the elevation remains finite, and the increase in the wave slope is also found to remain finite.† This suggests that the accumulation of the nonlinearity in the edge

† For an imperfect reflection, it happens that the elevation and wave slope increase considerably, which may eventually lead to wave breaking. This tendency is found for waves of relatively large amplitude (elevation) over a very gentle beach slope (Peregrine 1967; Pedersen & Gjevik 1983).

layer is still negligible, which is naturally understood from the assumption that the width of the edge layer is comparable to one wavelength of the incident waves.

To investigate the behaviour of solitary waves on such a gentle beach as $\xi_s^{-1} = O(\beta^{\frac{1}{2}})$, there may be an alternative way to employ the simplified shallow-water equation on an uneven bottom. Indeed, Mei & Le Méhauté (1966) and Peregrine (1967) derived a system of dispersive shallow-water equations which takes account of the gentle variation of the bottom surface $O(\beta^{\frac{1}{2}})$. Also Kakutani (1971) and Johnson (1973) derived a Korteweg–de Vries equation with variable coefficients for a far gentler uneven bottom corresponding to $\xi_s^{-1} = O(\beta^{\frac{3}{2}})$. Among them, Peregrine (1967) investigated the climb-up behaviour of a solitary wave based on his system of equations. But the essential difference from the present theory lies in the fact that his system does not take account of the existence of a shoreline, and therefore it is inapplicable up to there. In other words, his system describes an intermediate region between the offshore region of constant depth and a vicinity of the shoreline. In the present analysis, on the other hand, it is emphasized that the existence of a shoreline is explicitly assumed.

5. Edge-layer solution for a plane beach

So far we have been concerned with the derivation of the 'reduced' boundary condition. Here we demonstrate how to obtain an edge layer solution explicitly. Since it is difficult to obtain a solution for a general beach surface, we now restrict ourselves to a plane beach with constant inclination $\tan\theta = O(1)$; that is, $\xi = b(z) = -z\cot\theta$.

The edge-layer solution to (3.8) under the boundary conditions (3.9), (3.10*a*), (3.12) and (4.4) is easily obtained by the Green-function method. Let $G(\xi', z'; \xi, z)$ be the Green function defined in the semi-infinite trapezoidal domain S' (see figure 2):

$$G_{\xi'\xi'} + G_{z'z'} = \delta(\xi'-\xi, z'-z), \tag{5.1}$$

with $G_{n'} = 0$ on $\partial S'$, where δ denotes the delta function and $G_{n'}$ denotes the outward normal derivative to the boundary $\partial S'$. Here it should be remarked that $G_{n'} \neq 0$ at the infinite open end, because an integration of (5.1) over the whole domain S' yields unity. But using (3.12) in applying Green's formula, the solution to (3.8) is expressed as

$$\psi(\xi, z) = \int_0^1 G(-z'\cot\theta, z'; \xi, z)\,(z'-\tfrac{1}{2})\,\mathrm{d}z'. \tag{5.2}$$

Thus solving (3.8) is reduced to finding the Green function. To find it, a sophisticated method of conformal mapping provides a powerful tool. Invoking the well-known theorem of Schwarz and Christoffel (e.g. Milne-Thomson 1962), we transform the domain S' into the semi-infinite upper plane of ζ' with correspondence of the vertices A_2 and A_3 in the Z' plane to $\zeta' = -1$ and $\zeta' = 1$ respectively (see figures 2 and 3). Upon defining the complex variable $Z' = \xi' + \mathrm{i}z'$, the solution to the following differential equation provides such a transformation from the Z' plane to the complex ζ' plane:

$$\frac{\mathrm{d}Z'}{\mathrm{d}\zeta'} = C_1(\zeta'+1)^{\theta/\pi-1}(\zeta'-1)^{-\theta/\pi}, \tag{5.3}$$

where C_1 is a constant to be determined later. Integration of (5.3) is rather difficult for an arbitrary θ. However, it can be solved analytically if we assume that $\theta = \pi/m$, $m(\geqq 2)$ being an integer. To do so, we first introduce a complex-parametric plane w':

$$w' = \left(\frac{\zeta'+1}{\zeta'-1}\right)^{1/m}. \tag{5.4}$$

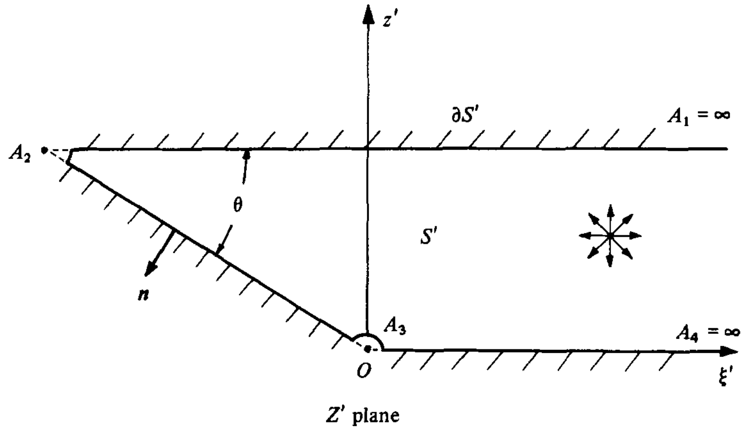

FIGURE 2. Complex Z' plane; several special points are designated by A_i ($i=1,2,3,4$): $A_1 = \infty$, $A_2 = -\cot\theta + \mathrm{i}$, $A_3 = 0$ and $A_4 = \infty$; for explicit indication of the singular points A_2 and A_3 in (5.3), the semi-infinite trapezoidal region is indented by the small circular arcs; the inside domain is designated by S', while its boundary is $\partial S'$.

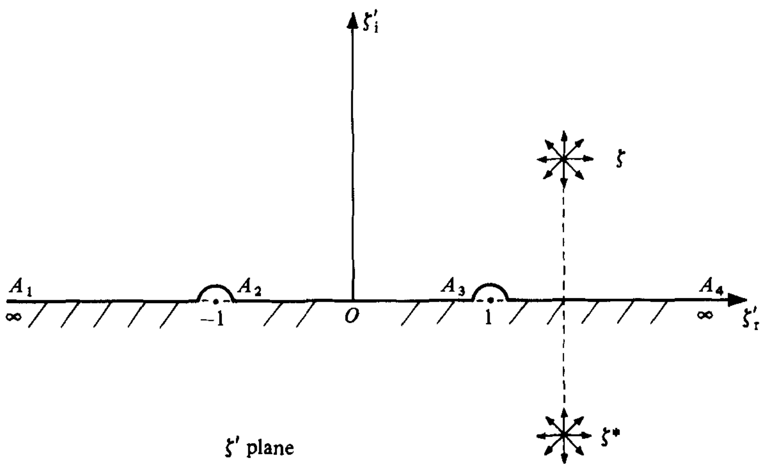

FIGURE 3. Complex ζ' plane; each A_i ($i = 1,2,3,4$) corresponds to the respective point indicated in figure 2: $A_1 = A_4 = \infty$, $A_2 = -1$ and $A_3 = 1$.

In this w' plane the upper plane of ζ' is transformed into the wedge region (see figure 4). Rewriting (5.3) in terms of w', it is reduced to

$$\frac{\mathrm{d}Z'}{\mathrm{d}w'} = -\frac{C_1 m}{w'^m - 1}, \tag{5.5}$$

whose solution is easily obtained as

$$Z' = -C_1 \sum_{j=0}^{m-1} w_j \log(w' - w_j) + C_2, \tag{5.6}$$

where $w_j (j=0,1,\dots,m-1)$ are the m roots of the equation $w'^m = 1$, that is, $w_j = \exp(2\pi \mathrm{i} j/m) = \exp(2\mathrm{i}\theta j)$, and C_2 is an arbitrary constant as well as C_1. Here and henceforth the principal value of the logarithmic function is defined by taking the range, $-\pi \leqq \arg\log(\dots) < \pi$. To complete the solution (5.6), the two constants C_1 and C_2 should be determined by specifying the correspondence of two special points

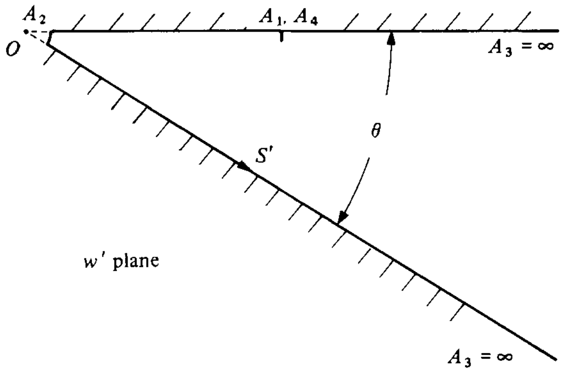

FIGURE 4. Complex w' plane; each A_i ($i = 1, 2, 3, 4$) corresponds to the respective point indicated in figure 2: $A_1 = A_4 = 1$, $A_2 = 0$ and $A_3 = \infty$.

A_2 and A_3 in the Z' and ζ' planes. In applying the theorem of Schwarz and Christoffel, A_2 is already assumed to correspond to $\zeta' = -1$, and therefore $w' = 0$, while A_3 corresponds to $\zeta' = 1$ and $w' = \infty$. Considering first A_3, we substitute $Z' = 0$ and $w' = \infty$ into (5.6). Noticing that $\Sigma_{j=0}^{m-1} w_j = 0$, C_2 must be chosen to be zero. Next for A_2, substituting $Z' = -\cot\theta + \mathrm{i}$ and $w' = 0$ in (5.6) and paying attention to taking the principal value, it follows that $C_1 = 1/\pi$, where the relation $\Sigma_{j=0}^{m-1} jw_j = -\frac{1}{2}m(1+\mathrm{i}\cot\theta)$ has been used. Thus two constants are determined as

$$C_1 = 1/\pi, \quad C_2 = 0. \tag{5.7}$$

Hence the mapping from Z' to ζ' via w' has been completed. The correspondence of several special points in the three planes is indicated in figures 2–4, where the boundary, of course, corresponds to the boundary.

After such a preparation, let us now seek the Green function G. From the standpoint of the fluid-dynamical interpretation, G represents the velocity potential for the flow field caused by a source of strength $1/2\pi$ placed at $\xi' = \xi$ and $z' = z$ in S'. To obtain G, it is convenient to introduce the complex velocity potential $F(Z')$ defined by

$$F(Z' = \xi' + \mathrm{i}z') = G(\xi', z') + \mathrm{i}H(\xi', z'), \tag{5.8}$$

where $H(\xi', z')$ is the stream function. From the theory of conformal mapping it is well known that the mapping does not destroy the harmonic properties of a function and that a streamline in the Z' plane corresponds to the same one in the ζ' plane. Also it is known that the mapping brings the source at $Z' = Z = \xi + \mathrm{i}z$ into that at $\zeta' = \zeta$ with its strength conserved. Thus the explicit form of F in the ζ' plane can easily be obtained by using the mirror-image principle:

$$F(\zeta'; \zeta, \zeta^*) = \frac{1}{2\pi}\log(\zeta' - \zeta) + \frac{1}{2\pi}\log(\zeta' - \zeta^*), \tag{5.9}$$

where ζ^* denotes the complex conjugate of ζ. The Green function in the ζ' plane is obtained by taking the real part of (5.9):

$$G(\zeta_r', \zeta_i'; \zeta_r, \zeta_i) = \frac{1}{2\pi}\log|(\zeta' - \zeta)(\zeta' - \zeta^*)|, \tag{5.10}$$

where we have set $\zeta' = \zeta_r' + \mathrm{i}\zeta_i'$ and $\zeta = \zeta_r + \mathrm{i}\zeta_i$. The Green function $G(\xi', z'; \xi, Z)$ in (5.2) would be derived if the transformation (5.4) and (5.6) with (5.7) were inversely solved to express ζ' in terms of Z' only. Because of the simple form of (5.10), however, it is rather advantageous to employ not $G(\xi', z'; \xi, z)$ but (5.10) itself in evaluating (5.2). To this end, we transform the integration (5.2) along the beach A_3A_2 in the Z' plane to that in the ζ' plane. In the w' plane, A_3A_2 is given by (see figure 4)

$$w' = s'\mathrm{e}^{-\mathrm{i}\theta}, \tag{5.11}$$

where s' is real ($0 < s' < \infty$). Substituting this into (5.6), and setting $Z' = (-\cot\theta + \mathrm{i})\,z'$, the relation between z' and s' is obtained as

$$z' = \frac{1}{\pi}\sin\theta \sum_{j=0}^{m-1} \tilde{w}_j \log(s' - \tilde{w}_j), \tag{5.12}$$

where $\tilde{w}_j (j = 0, 1, 2, \ldots, m-1)$ are defined by $\tilde{w}_j = w_j \exp(\mathrm{i}\theta)$. In the ζ' plane, on the other hand, A_3A_2 corresponds to the segment along the real axis from $\zeta' = 1$ to $\zeta' = -1$. Setting $\zeta' = \zeta'_{\mathrm{r}}$, it follows from (5.4) with (5.11) that

$$s' = \left(\frac{1+\zeta'_{\mathrm{r}}}{1-\zeta'_{\mathrm{r}}}\right)^{1/m}. \tag{5.13}$$

Eliminating s' in (5.12) by using (5.13), z' is expressed in terms of ζ'_{r}. Using $z' = z'(\zeta'_{\mathrm{r}})$ thus derived, (5.2) is transformed into an integration with respect to ζ'_{r}:

$$\psi(\xi, z) = -\int_{-1}^{1} G(\zeta'_{\mathrm{r}}, 0; \zeta_{\mathrm{r}}, \zeta_{\mathrm{i}})\,[z'(\zeta'_{\mathrm{r}}) - \tfrac{1}{2}]\frac{\mathrm{d}z'}{\mathrm{d}\zeta'_{\mathrm{r}}}\mathrm{d}\zeta'_{\mathrm{r}}, \tag{5.14a}$$

with

$$\frac{\mathrm{d}z'}{\mathrm{d}\zeta'_{\mathrm{r}}} = -\frac{1}{\pi}\sin\theta(1-\zeta'_{\mathrm{r}})^{-1/m}(1+\zeta'_{\mathrm{r}})^{1/m-1}, \tag{5.14b}$$

where (5.14b) is obtained directly from (5.3) with $\theta/\pi = 1/m$ and $C_1 = 1/\pi$ by setting $Z' = -\exp(-\mathrm{i}\theta)z'/\sin\theta$ and $\zeta' = \zeta'_{\mathrm{r}}$. Thus ψ can be expressed analytically in an integral form. It is difficult, however, to carry out the integration analytically, so that we must evaluate the integral by a numerical method. This will be studied in a forthcoming paper.

Finally we shall confirm whether the matching condition (3.12) is satisfied by the solution (5.14a), although the boundary condition (4.4) derived from mass conservation guarantees it equivalently. To examine this, we seek an asymptotic solution of ψ as $\xi \to \infty$ and $0 < z < 1$. The infinity $\xi \to \infty$ in the Z' plane corresponds to the infinity $\zeta = \zeta_{\mathrm{r}} + \mathrm{i}\zeta_{\mathrm{i}} \to \infty$ in the ζ' plane ($\zeta_{\mathrm{i}} > 0$), and it also corresponds to the neighbourhood of $w' = 1$ in the w' plane. Putting $\zeta = \zeta_{\mathrm{r}} + \mathrm{i}\zeta_{\mathrm{i}} = \rho\exp(\mathrm{i}\sigma)$ ($\rho \gg 1$, $0 < \sigma < \pi$) in (5.4), w is given approximately by

$$w \sim 1 + \frac{2}{m\rho}\mathrm{e}^{-\mathrm{i}\sigma} + O(\rho^{-2}). \tag{5.15}$$

Substituting this in (5.6) with (5.7), it follows that

$$Z = \xi + \mathrm{i}z \sim -\frac{1}{\pi}\sum_{j=0}^{m-1} w_j \log\left(1 + \frac{2}{m\rho}\mathrm{e}^{-\mathrm{i}\sigma} - w_j\right) \sim -\frac{1}{\pi}\log\left(\frac{2\kappa}{m\rho}\mathrm{e}^{-\mathrm{i}\sigma}\right) + O(\rho^{-1}), \tag{5.16}$$

where the real number κ is defined by $\log\kappa = \Sigma_{j=1}^{m-1} w_j \log(1-w_j)$. From this, the asymptotic correspondence between Z and ζ as $Z \to \infty$ is established:

$$\rho = \frac{2\kappa}{m}\mathrm{e}^{\pi\xi}; \quad \sigma = \pi z, \tag{5.17}$$

where $\xi \to \infty$ and $0 < z < 1$.

Next let us evaluate $\psi(\xi, z)$ as $\xi \to \infty$ by using (5.14a). Since $\zeta \to \infty$ as $\xi \to \infty$, G is approximated by

$$G(\zeta_r', 0; \zeta_r, \zeta_i) = \frac{1}{2\pi}\log(\zeta_r^2+\zeta_i^2) - \frac{1}{\pi}\frac{\zeta_r}{\zeta_r^2+\zeta_i^2}\zeta_r' + \dots$$
$$= \frac{1}{\pi}\log\rho - \frac{1}{\pi\rho}\zeta_r' \cos\sigma + O(\rho^{-2}), \qquad (5.18)$$

where the first term $(\log\rho)/\pi \sim \xi$ implies $G_\xi \sim 1$ as $\xi \to \infty$. From the symmetry properties of G with respect to its arguments, it also implies $G_{\xi'} \sim 1$ as $\xi' \to \infty$, which is consistent with the previous remark that $G_{n'} \neq 0$ at the infinite open end. Inserting (5.18) into (5.14a), and using (5.17), the asymptotic form of ψ as $\xi \to \infty$ is obtained as

$$\psi(\xi, z) \sim A\mathrm{e}^{-\pi\xi}\cos(\pi z) + O(\mathrm{e}^{-2\pi\xi}), \qquad (5.19a)$$

with
$$A = \frac{m}{2\pi\kappa}\int_{-1}^{1} \zeta_r'[z'(\zeta_r') - \tfrac{1}{2}]\frac{\mathrm{d}z'}{\mathrm{d}\zeta_r'}\mathrm{d}\zeta_r', \qquad (5.19b)$$

where we note that the first term of (5.18) does not contribute to (5.19a) because no total flux is assumed across $\xi' = -z'\cot\theta$. Hence it is verified that ψ decays exponentially as $\xi \to \infty$ and satisfies the matching condition (3.12). It should be remarked here that (5.19a) corresponds to the evanescent mode. Finally we note that for a gentle slope $O(\beta^{\frac{1}{2}})$, the edge layer solution is obtained on replacing $z' - \frac{1}{2}$ in (5.2) by (4.8) divided by $2f_x$.

6. Concluding remarks

In this paper the edge-layer theory has been developed for the sloping beach to derive the 'reduced' boundary condition (4.3) relevant to the shallow-water equation (2.6). As far as (4.3) is concerned, however, it can also be derived intuitively without any introduction of an edge layer. Noting that the beach surface is given by $x = \beta^{\frac{1}{2}}b(z)$, we apply the boundary condition at the beach surface directly to ϕ given in (2.4); that is, $\phi_x - \beta^{-\frac{1}{2}}b_z\phi_z = 0$ at $x = \beta^{\frac{1}{2}}b(z)$. Expanding the condition at $x = \beta^{\frac{1}{2}}b(z)$ around $x = 0$, one obtains

$$f_x + \beta^{\frac{1}{2}}f_{xx}\frac{\mathrm{d}}{\mathrm{d}z}(zb) + \tfrac{1}{2}\beta f_{xxx}\left[\frac{\mathrm{d}}{\mathrm{d}z}(zb^2) - z^2\right] + O(\beta^{\frac{3}{2}}) = 0 \quad \text{at} \quad x = 0. \qquad (6.1)$$

Integrating (6.1) over the depth and noting that $[f_{xxx}]_{x=0} = O(\beta^{\frac{1}{2}})$ from (4.9) and (6.1), we recover (4.3). Therefore it is found that the 'reduced' boundary condition is derived by averaging the boundary condition at the beach surface. But the important point is that the edge-layer theory can not only justify the relevance of the averaging but can also provide a correct description of the near-shore behaviour.

Last but not least, it should be remarked that the presence of the sloping beach gives rise to a correction $O(\beta^{\frac{1}{2}})$ to the shallow-water waves through the 'reduced' boundary condition (4.3), and even a correction $O(1)$ through (4.13) when the beach slope is as gentle as $O(\beta^{\frac{1}{2}})$. This is to be compared with the case of the vertical wall, for which the corresponding boundary condition is simply given by $f_x = 0$ and the effect of the wall remains to give only a phase shift $O(\beta)$ (Oikawa & Yajima 1973; Miles 1977a; Funakoshi & Oikawa 1982).

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